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MULTIPLE CRITERIA IN OPTIMAL CONTROL

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College of Commerce and Business Administration
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# MULTIPLE CRITERIA IN OPTIMAL CONTROL

Ronald J. Stern*

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## Abstract

In this paper we generalize the structure set forth in a multicriteria mathematical programming setting (see, e.g. [3], [4]) to the case of the optimal control of a linear system with more than one quadratic criterion. Procedures for approximating efficient points and solutions to bicriterion control problems are given for cases with and without magnitude restraints.

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# MULTIPLE CRITERIA IN OPTIMAL CONTROL

1. Introduction. Consider the following linear system of m ordinary differential equations:

$$\dot{x} = A(t)\,x + B(t)u \qquad (t_o \leq t \leq T_o) \tag{1.1}$$

with initial condition

$$x(t_o) = x_o \tag{1.2}$$

Here A(t) and B(t) are continuous (m x m) and (s x m) matrices on the interval $[t_o, T_o]$, respectively. The control u is any integrable $R^s$ - valued function on $[t_o, T_o]$, and for each u a unique solution to (1.1) - (1.2) is determined by

$$x(t) = S(t,t_o)x_o + \int_{t_o}^{t} S(t,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma \tag{1.3}$$

where $S(t,\sigma)$ is the fundamental solution of $\dot{x} = A(t)x$.

We now introduce n "quadratic" criterion functions:

$$\begin{aligned} f_i(u) = {} & \left\langle x(T_o) - \xi_i,\ W_i\,[x(T_o) - \xi_i] \right\rangle \\ & + \int_{t_o}^{T_o} \left\langle \tilde{x}_i(t) - C_i(t)\,x(t),\ Q_i(t)\,[\tilde{x}_i(t) - C_i(t)\,x(t)] \right\rangle dt \\ & - \int_{t_o}^{T_o} \left\langle u(t),\ R_i(t)\,u(t) \right\rangle dt \qquad (1 \leq i \leq n). \end{aligned} \tag{1.4}$$

The $\xi_i$ are fixed vectors in $R^m$; $\tilde{x}_i(t)$ are continuous $R^m$ - valued functions on $[t_o, T_1]$; $W_i$ are constant symmetric (m x m) matrices; $R_i(t)$, $Q_i(t)$ and $C_i(t)$ are continuous matrices on $[t_o, T_o]$; $Q_i(t)$ and $R_i(t)$ are symmetric, and $R_i(t)$ is positive definite on $[t_o, T_1]$. Here $T_1 > T_o$.

By an efficient point over a class of controls $\mathcal{C}$, we shall mean a control $u^o \in \mathcal{C}$ such that for no other $u^1 \in \mathcal{C}$ do we have $f_i(u^1) \geq f_i(u^o)$, $i = 1,2,\ldots,n$, with at least one inequality holding strictly. We now define two efficient point problems.

$E_1$ Find all efficient points over the class $L^{2,s}(t_o,T_o)$.
($L^{2,s}(t_o,T_o)$ is the space of controls such that $\int_{t_o}^{T_o} |u(t)|^2 dt < \infty$, where $|\ |$ denotes the Euclidean norm.)

$E_2$ Find all efficient points over the class of measurable controls for which $|u(t)| \leq 1$ almost everywhere on $[t_o,T_o]$.

For the case $n = 2$ we will study two other control problems, $M_1$ and $M_2$, related to the above efficient point problems.

$M_1$ Maximize $\min\{f_1(u), f_2(u)\}$
subject to $u \in L^{2,s}(t_o,T_o)$.

Let $h:R^2 \to R^1$ be a function which is continuous and non-decreasing in each of its arguments on the non-negative orthant $R^2_+$. Furthermore, assume that h is quasiconcave over $\text{int}(R^2_+)$. (Notice that the objective function in $M_1$ is of this type, and in fact possesses all the above properties on the whole space $R^2$.) Assume that $|u(t)| \leq 1$ almost everywhere on $[t_o,T_o]$ implies $f_i(u) > 0$, $i = 1,2$. (This is guaranteed, for example, if $W_i$ are positive definite and $|x_o|$ is sufficiently large.)

$M_2$ Maximize $h[f_i(u), f_2(u)]$
subject to $u \in \mathcal{U}$,

where $\mathcal{U}$ is the class of measurable controls u such that $|u(t)| \leq 1$ almost everywhere on $[t_o, T_o]$.

Let us remark here that the assumptions on the positivity of $f_1$ and $f_2$ were made since several interesting types of h are both monotone and quasiconcave only over $\text{int}(R^2_+)$; e.g., $h(x,y) = x^\beta y$ for $\beta > 0$, and $c_1 x^{\beta_1} + c_2 y^{\beta_2}$ for $c_1, c_2, \beta_1, \beta_2 > 0$.

In section 2, a simple variational method is given for solving $E_1$. In section 3, a penalty function approach, combined with a similar variational method, is employed to solve $E_2$. In section 4, we use the results of the previous sections to outline procedures for computing approximate solutions of $M_1$ and $M_2$.

2. Solution of $E_1$. We require the following lemma.

Lemma 2.1 *There exists a positive constant* N *such that if* $T_o - t_o \leq N$, *then each* $f_i$ *is strictly concave over the space* $L^{2,s}(t_o, T_o)$ *and is bounded above.*

Proof. The convavity is proven by verifying negativity of the second Gateaux derivatives of the $f_i$. The proof is a computation which makes use of Hölder's inequality. Details may be found in [2], section 8.4. The fact that the $f_i$ are bounded above is proven similarly, and this too is a straightforward application of Hölder's inequality. Details are omitted.

Let $\alpha_i \geq 0$, $1 \leq i \leq n$ be such that $\sum_{i=1}^{n} \alpha_i = 1$. Write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We now define another problem:

$P_1^\alpha$ Maximize $\sum_{i=1}^{n} \alpha_i f_i(u)$

subject to $u \in L^{2,s}(t_o, T_o)$

Theorem 2.1. There exists a positive constant $N^1$ such that if $T_o - t_o \leq N^1$, then for each $\alpha$ as above the problem $P_1^\alpha$ has a unique solution.

Proof. Let $f^\alpha = \sum_{i=1}^{n} \alpha_i f_i$. In view of Lemma 2.1, if $T_o - t_o \leq N$, then the condition

$$(2.1) \qquad \frac{d}{d\varepsilon} f^\alpha (u^\alpha + \varepsilon v) = 0 \quad \text{at } \varepsilon = 0 \text{ for any } v \in L^{2,s}(t_o, T_o)$$

is necessary and sufficient for $u^\alpha$ to be the unique solution of $P_1^\alpha$. Using the symmetry of the $Q_i(t)$ and $R_i(t)$, (2.1) yields the following linear integral equation for $u^\alpha$:

$$(2.2) \qquad 2u^\alpha(s) = \left[\sum_{i=1}^{n} \alpha_i R_i(s)\right]^{-1} \sum_{i=1}^{n} \alpha_i \Big\{ B^*(s)\, S^*(T_o, s)\, W_i \left[x^\alpha(T_o) - f_i\right] + \int_s^{T_o} B^*(s)\, S^*(t,s)\, C_i^*(t)\, Q_i(t) \left[\tilde{x}_i(t) - C_i(t)\, x^\alpha(t)\right] dt \Big\}$$

where $x^\alpha(t)$ is the solution to (1.1) - (1.2) corresponding to $u^\alpha(t)$. Using (1.3), we see that (2.2) can be written in the form:

$$(2.3) \qquad u^\alpha = A^\alpha u^\alpha$$

Here $A^\alpha$ is an operator taking $C^{o,s}(t_o, T_o)$, the space of continuous $R^s$ - valued functions endowed with the sup-norm, into itself. By comparing (2.3) with similar equations occurring e.g., in [2], we have the existence of $N^1 > 0$ such that $T_o - t_o \leq N^1$ guarantees that each possible $A^\alpha$ in (2.3) is a contraction.

We now can prove the main result of this section.

Theorem 2.2. Let $T_o - t_o \leq \min\{N, N^1\}$. Then

(i) If $\alpha_i > o$ for every $i = 1,2,\ldots,n$ then the solution $u^\alpha$ of $P_1^\alpha$ is an efficient point over $L^{2,s}(t_o,T_o)$.

(ii) If $u^o$ is an efficient point over $L^{2,s}(t_o,T_o)$ then for some $\alpha^o$ with non-negative components we have $u^o = u^{\alpha^o}$.

Proof. (i) is obvious. The proof of (ii) is as in the finite dimensional case, to be found in [4], Section 7.4

Remark. A method for approximating a solution of $P_1^\alpha$ is suggested by Theorem 2.1 -- namely, $u^\alpha$ in (2.3) is the uniform limit of a sequence of successive approximations. Another approach is also available for solving $P_1^\alpha$, namely the "synthesis" or "feedback" approach. (See, e.g., [2] Sect. 8.6 or [5] Sect. 5.2) Here one encounters a matrix differential equation of Riccati type whose solution is used to express $u^\alpha$. One must, in general, resort to numerical methods in order to approximately solve the Riccati equation.

3. Solution of $E_2$. For $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ with non-negative components summing to 1, we define the following problem:

$P_2^\alpha$ Maximize $\sum_{i=1}^{R} \alpha_i f_i(u) = f^\alpha(u)$

subject to $u \in \mathcal{U}$

Since $\mathcal{U}$ is not compact in the topology of $L^{2,s}(t_o,T_o)$, we require the following result on the consistency of the family $\{P_2^\alpha\}$.

Theorem 3.1. Let $T_o - t_o \leqq N$ where N is as in Lemma 2.1. Then each $P_2^\alpha$ has a unique solution.

Proof. By the strict concavity of the objective function $f^\alpha$ and the fact that $\mathcal{U}$ is a convex set, we need only to prove existence of a solution for each $P_2^\alpha$. To this end, let $\{u_n\}$ be a sequence in $\mathcal{U}$ such that:

$$f^\alpha(u_n^\alpha) \longrightarrow \sup_{u \in \mathcal{U}} f^\alpha(u) = c^\alpha$$

It is clear that $\sup_\alpha c^\alpha < \infty$. Since $\{u_n^\alpha\}$ is a bounded sequence in $L^{2,s}(t_o,T_o)$, we can extract a subsequence $\{u_{n'}^\alpha\}$ which converges weakly to $u_w^\alpha$. In [2] it is proven that $u_w^\alpha$ is a member of $\mathcal{U}$. Furthermore, it is shown there that $x_{n'}^\alpha \longrightarrow x_w^\alpha$ in $C^o, {}^m[t_o, T_o]$ where $x_{n'}^\alpha$ is the solution of (1.1) - (1.2) corresponding to $u_{n'}^\alpha$. It is known [5] that if $g(t,u)$ is continous in t and concave in u, then:

$$(3.1) \qquad \lim_{n' \to \infty} \sup \int_{t_o}^{T_o} g\left(t, u_{n'}^\alpha(t)\right) dt \leqq \int_{t_o}^{T_o} g\left(t, u_w^\alpha(t)\right) dt.$$

Identifying $g(t,u) = -\sum_{i=1}^{n} \alpha_i \left\langle u, R_i(t)u \right\rangle$, it follows that $f^\alpha(u_w^\alpha) = c^\alpha$.

The same proof as Theorem 2.2 yields the following:

Theorem 3.2. Let $T_o - t_o \leq N$ where N is as in Lemma 2.1. Then:

(i) If $\alpha_i > o$ for every $i = 1,2,\ldots,n$, then the solution of $P_2^\alpha$ is an efficient point over $\mathcal{U}$.

(ii) If $u^o$ is an efficient point over $\mathcal{U}$ then for some $\alpha^o$ with non-negative components $u^o$ solves $P_2^{\alpha^o}$.

We turn now to the problem of approximating the solution to $P_2^\alpha$. We remark here that the feedback approach is in general intractable for this problem. We will derive a computational scheme which resembles that given in [6]. The idea is to employ penalty functions of such a type that the approximating "penalty problem," which has no constraints, has a solution which solves a certain integral equation and can be identified as the limit of a certain iterative method.

Let $\rho(u)$ be the Euclidean distance of a vector $u \in R^s$ to the unit ball $\{u: |u| \leq 1\}$ in $R^s$. Let $o < \epsilon \leq 1$, and define the following function:

$$f_\epsilon^\alpha(u) = f^\alpha(u) - \frac{1}{\epsilon}\int_{t_o}^{T_o} \rho^2\left(u(t)\right)\,dt. \tag{3.2}$$

Let $c_\epsilon^\alpha = \sup_{u \in L^{2,s}(t_o,T_o)} f_\epsilon^\alpha(u)$, which clearly is finite. An argument which uses (3.1) can be used to show that $c_\epsilon^\alpha$ is uniquely attained, but this will not be required. We now will prove the following theorem:

Theorem 3.3. Let $T_o - t_o \leq N$. Then $c_\epsilon^\alpha$ decreases to $c^\alpha$ as $\epsilon \to o$.

Proof. That $\{c_\epsilon^\alpha\}$ is decreasing and that $c_\epsilon^\alpha \geq c^\alpha$ for each $0 < \epsilon \leq 1$ are obvious. Let W be an upper bound over $L^{2,s}(t_o,T_o)$ of $f^\alpha(u)$, which is finite by Lemma 2.1. Define, for any positive number R,

$$S[R] = \left\{u \in L^{2,s}(t_o,T_o): \int_{t_o}^{T_o} \varrho^2\left(u(t)\right) dt \leq R\right\}$$

It follows that

$$(3.3) \qquad c_\epsilon^\alpha = \sup_{u \in S[\epsilon(W-c_\alpha)]} f_\epsilon^\alpha(u)$$

To prove the assertion of the theorem, it will suffice to determine a constant $F^\alpha$ such that for given $u \in S[\epsilon(W-c_\alpha)]$ there exists $\hat{u} \in \mathcal{U}$ such that:

$$(3.4) \qquad f_\epsilon^\alpha(u) - f^\alpha(\hat{u}) \leq F^\alpha \epsilon^{1/2}.$$

We will identify $\hat{u}$ next.

Letting v be any vector in $R^s$, we define $\mathcal{U}(v)$ to be the subset of the unit ball such that $\varrho(v) = d(v,\mathcal{U}(v))$, the (Euclidean) distance between v and $\mathcal{U}(v)$. We now introduce a lexicographic ordering on $R^s$. Write:

$$(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s) \ll (v_1, v_2, \ldots, v_s)$$

if either $\hat{v}_1 < v_1$, or $\hat{v}_1 = v_1, \ldots, \hat{v}_{q-1} = v_{q-1}, \hat{v}_q < v_q$ for some $q = 2,\ldots,s$. Thus, there is a unique point $\hat{v} \in \mathcal{U}(v)$ such that

$\hat{v} \ll w$ for all $w \in \mathcal{U}(v)$. The map $v \longrightarrow \bar{v}$ is therefore a single-valued correspondence taking $R^s$ onto the unit ball in $R^s$. By a result due to Fillipov ([1] or [2]), we have that $\hat{u}(t)$ is measurable, where $u(t)$ is the given member of $L^{2,s}(t_o,T_o)$. Let $\hat{x}(t)$ denote the solution to (1.1) - (1.2) corresponding to $\hat{u}(t)$, and let $x(t)$ correspond to $u(t)$. To show that (3.4) holds for some $F^\alpha$ is a straightforward calculation, which makes use of the fact that

$$(3.5) \qquad \int_{t_o}^{T_o} |\hat{u}(t) - u(t)|\, dt \leq \left(\varepsilon (W - c_\alpha)\right)^{1/2} (T_o - t_o)^{1/2},$$

which follows from the definition of $S[\varepsilon(W - c_\alpha)]$ and Holder's inequality.

Remark. If in the definition of $f_\varepsilon^\alpha$ we replace $\rho$ with any Borel measurable function $q$ which is 0 on the unit ball in $R^s$ and dominates $\rho$, then the assertion of Theorem 3.3 can be generalized in an obvious manner.

Let $k$ be a positive integer. Define:

$$U_k = \{u \in R^s: \ |u| \leq k^{-1/2k}\}$$

We introduce the following function on $L^{2,s}(t_o,T_o)$:

$$(3.6) \qquad f_k^\alpha(u) = f^\alpha(u) - \int_{t_o}^{T_o} |u(t)|^{2K} dt$$

Define $\mathcal{U}_k$ to be the class of those measurable functions on $[t_o,T_o]$ valued almost everywhere in $U_k$. Define next:

$$\tilde{c}_k^{\alpha} = \sup_{u \in \mathcal{U}_k} f_k^{\alpha}(u)$$

and

$$c_k^{\alpha} = \sup_{u \in L^{2,s}(t_o,T_o)} f_k^{\alpha}(u)$$

We have the following result:

Theorem 3.4. $c_k^{\alpha}$ converges to $c^{\alpha}$ as $k \to \infty$.

Proof. The proof consists of two parts, namely:

(i) $\tilde{c}_k^{\alpha} \to c^{\alpha}$ as $k \to \infty$.

(ii) $|c_k^{\alpha} - \tilde{c}_k^{\alpha}| \to 0$ as $k \to \infty$.

(i) is proven as follows: Given $u \in \mathcal{U}$ define $u_k \in \mathcal{U}_k$ by $k^{-1/2k}u$, and compute a constant $D^{\alpha}$ such that:

$$|f^{\alpha}(u) - f^{\alpha}(k^{-1/2k}u)| \leq D^{\alpha}[1-k^{-1/2k}].$$

We now will verify (ii). Define:

$$\phi_k(u) = \begin{cases} 0 & \text{if } u \in U_k \\ |u|^{2k} & \text{if } u \notin U_k \end{cases}$$

and define the following function:

$$f_{\phi_k}^{\alpha}(u) = f^{\alpha}(u) - \int_{t_o}^{T_o} \phi_k(u)\, dt \tag{3.7}$$

Since $k^{1/2} \cdot d(u, U_k) \leq \phi_k(u)$, we can apply the remark following Theorem 3.3 and obtain:

(3.8) $$\left| \tilde{\tilde{c}}_k^{\alpha} \longrightarrow \tilde{c}^{\alpha} \right| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,$$

where

$$\tilde{\tilde{c}}_k^{\alpha} = \sup_{u \in L^{2,s}(t_o,T_o)} f_{\phi_k}^{\alpha}(u)$$

It is not difficult to see that

(3.9) $$\left| f_{\phi_k}^{\alpha}(u) - f_k^{\alpha}(u) \right| \leq \frac{(T_o - t_o)}{k} \quad \text{for all } u \in L^{2,s}(t_o,T_o).$$

(3.8) and (3.9) give (ii), completing the proof.

From now on consider the special case $R_i(t) = I$ for $1 \leq i \leq n$. This enables us to prove the following result:

Theorem 3.5. If $T_o - t_o \leq \min \{N, N^1\}$ then for each $\alpha$ and $k$ there is a unique element $u_k^{\alpha}$ of $L^{2,s}(t_o,T_o)$ at which the supremum of $f_k^{\alpha}$ is attained. Furthermore,

(3.10) $$f_i(u_k^{\alpha}) \longrightarrow f_i(u^{\alpha}), \quad i = 1,2,\ldots,n,$$

where $u^{\alpha}$ is the unique element at which the supremum over $\mathcal{U}$ of $f^{\alpha}$ is attained.

Proof. By concavity, existence implies uniqueness. The condition of stationarity, namely:

(3.11) $\frac{d}{d\varepsilon} f_k^\alpha (u^\alpha + \varepsilon v) = 0$ at $\varepsilon = 0$ for any $v \in L^{2,s}(t_o,T_o)$

yields the following integral equation for $u_k^\alpha$:

(3.12) $2\, u_k^\alpha(s) + 2k \mid u_k^\alpha(s) \mid^{2k-2} u_k^\alpha(s) =$

$$\sum_{i=1}^{n} \alpha_i \left\{ B^*(s)\, S^*(T_o,s)\, W_i \left[ x_k^\alpha(T_o) - \xi_i \right] + \int_s^{T_o} B^*(s)\, S^*(t,s)\, C_i^*(t)\, Q_i(t) \left[ \tilde{x}(t) - C_i(t)\, x_k^\alpha(t) \right] dt \right\}$$

where $x_k^\alpha$ is the solution to (1.1) - (1.2) which corresponds to $u_k^\alpha$.

A map $\mathcal{M}_k$: $R^s \longrightarrow R^s$ is defined by :

(3.13) $\mathcal{M}_k(u) = 2u + 2k \mid u \mid^{2k-2} u$

This transformation has an inverse given by:

(3.14) $\mathcal{M}_k^{-1}(w) = \dfrac{w}{2 + 2k\, [r_k(|w|)]^{2k-2}}$

where $r_k(|w|)$ is the unique real root of the polynomial $2kx^{2k-1} + 2x - |w|$. Since $\mathcal{M}_k$ is invertible, we can rewrite (3.12) as follows:

(3.15) $\mathcal{M}_k \left( u_k^\alpha(t) \right) = T_k^\alpha \left[ \mathcal{M}_k (u_k^\alpha) \right]$,

where $T_k^\alpha$ is an operator taking $C^{o,s}(t_o,T_o)$ into itself determined by (3.12) and (3.15). We will show that each $T_k^\alpha$ is a contraction when $T_o - t_o \leq N^1$. Since $A^\alpha$ in section 2 is a contraction under this assumption, we have, for each $\bar{u}_k^\alpha$ and $\bar{\bar{u}}_k^\alpha$ in $C^{o,s}(t_o,T_o)$:

(3.16) $\left\| T_k^\alpha \left[ \mathcal{M}_k(\bar{u}_k^\alpha) \right] - T_k^\alpha \left[ \mathcal{M}_k (\bar{\bar{u}}_k^\alpha) \right] \right\|$

$\leq \theta \left\| \bar{u}_k^\alpha - \bar{\bar{u}}_k^\alpha \right\|$, $0 < \theta < 1$, where $\| \ \|$ is the sup-norm in $C^{o,s}(t_o,T_o)$.

Thus, $T_k^\alpha$ contracts if:

$$(3.17) \qquad \left\| \tilde{u}_k^\alpha - \bar{\bar{u}}_k^\alpha \right\| \leq \left\| \mathcal{M}_k(\tilde{u}_k^\alpha) - \mathcal{M}_k(\bar{\bar{u}}_k^\alpha) \right\|$$

(3.17) follows from the fact that if $v_1$ and $v_2$ are in $R^s$, $|v_2| \geq |v_1|$, and $c_2 \geq c_1 \geq 1$, $c_1$ and $c_2$ real, then $|v_2 - v_1| \leq |c_2 v_2 - c_1 v_1|$.

It remains to prove (3.10). Define $\hat{u}_k^\alpha$ from $u_k^\alpha$ using the correspondence "^" as in the proof of Theorem 3.3. Using the arguments of the proof of Theorem 3.4, we have:

$$(3.18) \qquad f^\alpha(\hat{u}_k^\alpha) \longrightarrow f^\alpha(u^\alpha),\ 1 \leq i \leq n,\ \text{as } k \longrightarrow \infty,$$

and

$$(3.19) \qquad f_i^\alpha(\hat{u}_k^\alpha) \longrightarrow f_i^\alpha(u_k^\alpha) \text{ as } k \longrightarrow \infty.$$

From (3.18), the uniqueness of $u^\alpha$ as the minimizer of $f^\alpha$ over $\mathcal{U}$, and the fact that $\hat{u}_k^\alpha \in \mathcal{U}$, we must have:

$$(3.20) \qquad f_i^\alpha(\hat{u}_\alpha^k) \longrightarrow f_i^\alpha(u^\alpha),\ 1 \leq i \leq n \text{ as } k \longrightarrow \infty.$$

(3.19) and (3.20) give (3.10).

Remark. A successive approximation procedure for solving equations like (3.15) is found in [6], section 5. This procedure circumvents the fact that $r^k$ has no explicit form for $k \geq 2$. Details will not be given here.

4. Solution of $M_1$ and $M_2$. We first require the following lemma.

Lemma 4.1. If $T_o - t_o \leq N$ then $M_1$ and $M_2$ have optimal solutions, and at least one solution of each is efficient.

Proof. The proof will apply to both problems. Since each problem has a finite supremum, we can choose a sequence $u_n \in L^{2,s}(t_o,T_o)$ such that:

$$(4.1) \qquad h\left[ f_1(u_n),\ f_2(u_n) \right] \longrightarrow c$$

where h in the objective function for either problem and where c is the problem supremum. Since $\{u_n\}$ lies in a bounded subset of $L^{2,s}(t_o,T_o)$ for either problem, we can extract a weakly convergent subsequence $u_{n'}$ with weak limit $u_w$, which is feasible. By (3.1) and arguments preceding it, we must have $u_w$ as a problem maximizer, in view of the monotonicity of h in each variable. Remaining details may be found in [3], where (3.1) and the monotonicity of h in its arguments must again be employed.

In view of Lemma 4.1, we can state the following result, the proof of which is generalized from [3] (where, as for Lemma 4.1, it is given for a finite dimensional case). The proof makes use of the quasiconcavity of the objective functions h of $M_1$ and $M_2$, the assumptions made regarding $M_2$ in section 1, and the fact that n = 2.

Theorem 4.1. Let $T_o - t_o \leq N$. Then $h\left[ f_1(u^\alpha),\ f_2(u^\alpha) \right] = q(\alpha)$ is unimodal on $[0,1]$.

Since the location of suprema of unimodal functions can easily be accomplished by search techniques, Theorems, 2.1, 2.2, Lemma 4.1 and Theorem 4.1 constitute a procedure for approximately solving $M_1$, while Theorems 3.2, 3.5, Lemma 4.1 and Theorem 4.1 constitute a procedure for approximately solving $M_2$.

## REFERENCES


[1] I. Fillipov, "On Certain Questions in the Theory of Optimal Control," SIAM J. Control, 1, 1962, p. 76-84.

[2] A. Friedman, Differential Games, Wiley-Interscience, 1971.

[3] A. M. Geoffrion, "Solving Bicriterion Mathematical Programs," Operations Research, 15, 1967, p. 39-54.

[4] S. Karlin, Mathematical Methods and Theory in Games, Programming and Economics, Vol. I, Addison-Wesley, 1959.

[5] E. Lee and L. Markus, Foundations of Optimal Control Theory, Wiley, 1967.

[6] R. Stern, "Open Loop Approximation of Differential Games," SIAM J. Control, 11, 1973, xxx-xxx .

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